The area of the Mandelbrot boundary

Zero-area criteria for infinite satellite renormalization

Let $\mathcal{M} = \{c \in \mathbb{C} : (f_c^k(0))_{k\geq 0} \text{ is bounded}\}$, where $f_c(z) = z^2 + c$. All areas below are ordinary two-dimensional Lebesgue measures in the complex c-plane. Whether $\text{area}(\partial\mathcal{M}) > 0$ remains open. The at-most-finitely-renormalizable part of the boundary already has area zero [1]. Which infinite sequences of renormalizations could contribute area?

A class with exactly zero area

Consider nested little Mandelbrot copies obtained by successive satellite bifurcations. If the parent has period n_j and the limb has internal angle p_j/q_j in lowest terms, the child has period $n_{j+1} = n_j q_j$. At attachment the parent cycle has multiplier $e^{2\pi i p_j/q_j}$. The limb includes the bulb and all the features beyond it.

Theorem. The set of parameters in such infinite satellite towers for which

$$\limsup_{j\to\infty} \frac{\log q_j}{n_j} > 0$$

has planar area zero. Equivalently, some $\varepsilon > 0$ satisfies $q_j \geq e^{\varepsilon n_j}$ at infinitely many levels. The value of ε may depend on the parameter.

A second theorem on page 8 covers polynomial growth with small rotation angles. For example, $|p_j| = 1$ and $n_j^2 \leq q_j \leq 2n_j^2$ at every sufficiently late stage already imply zero area. Page 9 obtains Hausdorff dimension zero when the signed unit numerators are retained and $q_{j+1} = q_j^2$ eventually. These proofs cover whole families of itineraries.

The exponential-growth argument uses Levin's estimates for limb diameters and Lyubich's theorem on straightening little copies [2, 3]. Pages 2–3 construct the cover and page 4 transfers the result between copies. Levin obtained local connectivity under these growth conditions. Here the conclusion concerns area. Priority for this deduction has not been established.

Two numerical scales

All limbs attached directly to the main cardioid with $q \geq 1000$, taken together and including their interiors, have area less than 0.0044 by an analytic inequality (page 5). The contribution from the low-denominator limbs is still unknown.

For the *whole* boundary, a proof using exact coefficients and the two nonreal period-three components gives $\text{area}(\partial\mathcal{M}) < 0.85$. Two previously reported computer-assisted certificates give the sharper $\text{area}(\partial\mathcal{M}) < 0.31995$ by subtraction. Page 6 gives both derivations.

Counting possible parent boundaries

Let S_n be the union of the boundaries of all hyperbolic components of exact period n. A hyperbolic component W is a parameter region with an attracting cycle. Its multiplier map sends W conformally onto $\mathbb{D}$ and extends homeomorphically to the boundary. In particular, ∂W is connected, lies in $|c| \le 2$, and each of its parameters has a fixed point of f_c^n with multiplier of modulus one. Each component has its own center where $f_c^n(0) = 0$ [4]. Since this polynomial equation in c has degree 2^{n-1}, there are at most $h_n = 2^{n-1}$ such components.

Lemma 1. An algebraic curve containing S_n

There is a nonzero real polynomial $H_n(x, y)$ of total degree at most $D_n = n4^n$ whose zero set contains S_n, with $c = x + iy$.

Proof. Put $m = 2^n$. Form the multiplier polynomial

$$P_n(c, \lambda) = \prod_{f_c^n(z)=z} (\lambda - (f_c^n)'(z)),$$

where fixed points are counted with multiplicity. This is the resultant in z of the monic $f_c^n(z) - z$ and $\lambda - (f_c^n)'(z)$. It is a real-coefficient polynomial in c, λ, monic of degree m in λ.

Every periodic orbit lies in $|z| \le R_c = (1 + \sqrt{1 + 4|c|})/2$, since larger moduli escape under iteration. Hence $|(f_c^n)'(z)| \le (2R_c)^n$. The coefficient of λ^{m-k} in P_n is, up to sign, an elementary symmetric polynomial in these multipliers and has growth $O_n((1 + |c|)^{nk/2})$. Being a polynomial in c, its degree is at most $\lfloor nk/2 \rfloor$. Thus $\deg_c P_n \le nm/2$.

If there is a unit multiplier λ, then $P_n(c, \lambda)$ and $\lambda^m P_n(\bar{c}, 1/\lambda)$ share that root because $1/\lambda = \bar{\lambda}$. Take their resultant in λ and substitute $c = x+iy$, $\bar{c} = x-iy$. Each term in the Sylvester determinant contains at most m coefficient factors from each polynomial. Its total degree is at most $2m(nm/2) = n4^n$.

At $c = 6$, every periodic point has $|z| \le 3$ and $|z|^2 = |f_6(z) - 6| \ge 3$. All the multipliers therefore have modulus greater than one, whereas the roots of the reciprocal polynomial have modulus less than one. The resultant is real and nonzero there. Its real part provides the required H_n. □

What this counts

Except for lines on which H_n vanishes identically, a horizontal or vertical line meets the whole union S_n in at most D_n points. This bound holds regardless of how many different limbs attach to the parents.

From limb diameters to an area estimate

Lemma 2. For $0 < r \le 1$, the closed r-neighborhood of S_n has a cover by at most

$$48n4^n/r + 2^{n-1} \tag{1}$$

disks of radius $2r$.

Proof. Translate a square mesh of spacing r so that none of its horizontal or vertical lines is a line factor of H_n. At most $12/r$ of its lines meet $[-2,2]^2$. Each line meets S_n at most D_n times, with at most four adjacent closed squares per intersection. A further square meeting S_n but having no boundary intersection with S_n must contain an entire connected parent boundary. There are at most h_n such squares. A disk of radius $2r$ around a square's center covers the square and its closed r-neighborhood, since $1 + \sqrt{2}/2 < 2$. This proves (1). □

Limb estimates and the infinite cover

Write $L(W, p/q)$ for the limb, including its attachment point. Use the representative $-1/2 < p/q \le 1/2$, $p \neq 0$. Levin proves the following diameter bounds with absolute constants [2, Theorems 5–6].

$$\begin{aligned} &\operatorname{diam} L(W,p/q) \le C_4\, 4^n/|p| && \text{for every period-}n\text{ parent},\\ &\operatorname{diam} L(W,p/q) \le C_8\, 8^n/q && \text{for a nonprimitive period-}n\text{ parent}. \end{aligned} \tag{2}$$

A nonprimitive parent is itself a satellite component. Use $(K,u) = (4,|p|)$ or $(8,q)$ with the corresponding parent restriction. For $a > K$, let $E_K(a)$ denote parameters lying in a qualifying limb with $u \ge a^n$ for infinitely many distinct parent periods n. Each such limb at period n lies within $r_n = C_K(K/a)^n$ of S_n, because its attachment point lies in S_n.

By (1), for sufficiently large n the total s-dimensional cost of covering these limbs at period n is at most a constant times

$$n4^n r_n^{s-1} + 2^n r_n^s.$$

The series over n converges when $s > 1 + \log 4/\log(a/K)$, since $4(K/a)^{s-1} < 1$ and $2(K/a)^s < 1$. Its tails cover the limsup with arbitrarily small disks and arbitrarily small cost. Consequently

$$\boxed{\dim_H E_K(a) \le \min\left\{2, 1 + \frac{\log 4}{\log(a/K)}\right\}.} \tag{3}$$

For example, $\dim_H E_8(64) \le 5/3$, so it has area zero. The set satisfying $q \ge a^n$ infinitely often for *every* $a > 8$ at satellite parents has dimension at most one. These dimension bounds refer to the parameter coordinates in which the periods are counted.

Removing the fixed exponential threshold

For a direct satellite tower, fix one little copy of period $N = n_j$. Straightening this copy identifies it with the standard Mandelbrot set. A later parent of period n_k becomes a parent of period n_k/N, and its internal rotation fraction p_k/q_k is preserved. Beyond the first parent in these new coordinates, the parents in a direct satellite tower are again satellite parents. Taking the first-return map f_c^N accounts for the division of periods. Levin's proof gives the correspondence of rotation numbers and limbs [2, §3.2, (A2), (C2), E].

Proof of the theorem on page 1. For a parameter satisfying the hypothesis, choose $\varepsilon > 0$ for which $q_k \geq e^{\varepsilon n_k}$ infinitely often. Since $n_j \to \infty$, choose a *fixed* prefix with $\varepsilon N > \log 64$. At infinitely many later stages,

$$q_k \geq e^{\varepsilon n_k} > 64^{\,n_k/N}.$$

The straightened parameter belongs to $E_8(64)$, which has area zero by (3).

The inverse straightening of a primitive copy has a quasiconformal extension. For a satellite copy it has such extensions away from arbitrarily small neighborhoods of the root [3, Theorem 5.5 and the following remark]. Planar quasiconformal maps send area-zero sets to area-zero sets (Lusin's property N). A countable exhaustion away from the root, followed by the addition of that single point, therefore transfers nullity to each fixed copy. There are countably many finite prefixes, since there are finitely many centers of each period and countably many finite sequences of them. Their union is null, even though the choice of prefix and ε varies with the parameter. □

A version for numerators

More generally, consider a nested sequence of simple renormalizations of periods n_k. Write p_k/q_k for the rotation number of the separating fixed point of the kth renormalization, as in Levin's Theorem 1. If $\limsup_k \log|p_k|/n_k > 0$, the set of such parameters has area zero. Indeed, (A2) and (C2) in the same source place the straightened parameter in the appropriate p_k/q_k-limbs. Use $E_4(32)$ in place of $E_8(64)$. Equation (3) again gives dimension at most $5/3$ there. This numerator version allows primitive parents. The dimension bound applies before pullback, which preserves nullity.

Where the proof stops

Unbounded satellite denominators can grow subexponentially in the parent period. For example, the combinatorial choices $n_0 = 1$, $q_j = j + 2$ give $n_j = (j + 1)!$ and $\log q_j/n_j \to 0$. Removing any fixed prefix still leaves limit zero. The argument also leaves general primitive renormalizations with small numerators untreated. Pages 7–8 give separate geometric and arithmetic criteria that apply to some subexponential sequences.

Area in limbs attached to the main cardioid

Let T_Q be the union of all whole limbs attached directly to the main cardioid whose reduced denominators satisfy $q \geq Q$, with $Q \geq 2$. Subsequent dynamics in these limbs are unrestricted. Put $R = 4^{1/Q}$. Then

$$\boxed{\operatorname{area}(\partial\mathcal{M} \cap T_Q) \leq \operatorname{area}(T_Q) \leq \frac{\pi}{8}(R^4 + 2R^2 - 3).} \tag{4}$$

Proof. In the p/q-limb, a logarithm L of the separating fixed point's multiplier belongs to the closed Yoccoz disk with center $2\pi ip/q + (\log 2)/q$ and radius $(\log 2)/q$ [2, eq. (8) with $n = 1$]. Thus $0 \leq \Re L \leq 2\log 2/q$ and $1 \leq |\lambda| \leq R$ for $\lambda = e^L$. A fixed point with multiplier λ is $z = \lambda/2$, so its parameter is

$$c = h(\lambda) = \lambda/2 - \lambda^2/4.$$

All the high-denominator limbs are covered by the image of the *same* annulus $1 \leq |\lambda| \leq R$. Integrating the Jacobian counts preimages with multiplicity and gives an upper bound even where the map is two-to-one. Hence

$$\begin{aligned}\operatorname{area}(T_Q) &\leq \int_1^R \int_0^{2\pi} \tfrac{1}{4}|1 - re^{i\theta}|^2 r \, d\theta \, dr \\ &= \frac{\pi}{2}\int_1^R (r + r^3)\, dr = \frac{\pi}{8}(R^4 + 2R^2 - 3).\end{aligned}$$

The leading term of this upper bound is $2\pi \log 2/Q$. □

A hand check at denominator 1,000

Use $\log 2 < 0.694$, $\pi < 22/7$, and $e^x - 1 \leq x/(1-x)$ for $0 \leq x < 1$ (compare power series). In (4), $R^4 = \exp(8\log 2/1000)$ and $R^2 = \exp(4\log 2/1000)$. It follows directly that

$$\operatorname{area}(T_{1000}) < \frac{22}{56}\left(\frac{5552}{994448} + 2\frac{2776}{997224}\right) < 0.0044.$$

What this says about the full boundary

Distinct first-level limbs are disjoint up to possible attachment points. Every point of $\mathcal{M}$ outside the closed cardioid belongs to a limb [2, §2.3]. The cardioid boundary is a rectifiable curve of area zero. The sum of the boundary areas in the finitely many limbs with $q < Q$ therefore differs from $\operatorname{area}(\partial\mathcal{M})$ by at most (4). The unknown contributions inside those low-denominator limbs must still be controlled.

Two upper bounds for the whole boundary

An exact calculation with fifteen rational coefficients

The normalized exterior uniformization is $\Psi(w) = w + b_0 + \sum_{m\ge1} b_m w^{-m}$. Grönwall's area theorem gives $\operatorname{area}(\mathcal{M}) = \pi(1 - \sum_{m\ge1} m|b_m|^2)$ [5]. The main cardioid is $h(\mathbb{D})$ with $h(v) = v/2 - v^2/4$ univalent on $\mathbb{D}$, so its area is $3\pi/8$ by integration of $|h'|^2$. The period-two disk is $|c+1| < 1/4$ (its cycle multiplier is $4(c+1)$) and has area $\pi/16$. They have disjoint interiors. This gives

$$\operatorname{area}(\partial\mathcal{M}) \le \pi\left(\frac{9}{16} - \sum_{m=1}^{15} m|b_m|^2\right). \tag{5}$$

For the coefficients in (5), let $F_0(X) = X$, $F_{k+1}(X) = F_k(X)^2 + X$, and set $H(u) = u^{16}F_4(1/u)$. The Faber identity $F_4(\Psi(w)) = w^{16} + O(w^{-1})$ [5] implies that the inverse at infinity of $F_4(X)^{1/16} \sim X$ agrees with Ψ through w^{-15}. In $v = 1/w$, inversion of $u = vH(u)^{1/16}$ gives, by Lagrange inversion,

$$b_m = -\frac{1}{m}[u^{m+1}]H(u)^{m/16}, \qquad 1 \le m \le 15.$$

To check the expansion, take $H_0 = 1$, $H_{k+1} = H_k^2 + u^{2^{k+1}-1}$ and $H = H_4$. The integers $2^{2m+1}b_m$, in increasing order, are

$$1,\ -8,\ 15,\ 0,\ -94,\ -512,\ 987,\ 0,\ -7346,\ 65536,$$
$$-122058,\ 0,\ -2757820,\ -22020096,\ 59250963.$$

Consequently the sum in (5) is $1274974850201784103/2^{62} > 221/800$. Thus (5) is less than $229\pi/800$.

A further exclusion from the two nonreal period-three components

Their centers $\alpha, \bar\alpha$ are roots of $P(c) = c^3 + 2c^2 + c + 1$. The inverse multiplier branch $c(\lambda)$ satisfies $c^3 + 2c^2 + (1 - \lambda/8)c + (1 - \lambda/8)^2 = 0$ [9, §2.1]. Thus $c'(0) = (\alpha + 2)/(8P'(\alpha))$ at the first center. Integration of the squared derivative of a univalent disk map gives area at least $\pi|c'(0)|^2$ per component.

Let b be the real root. The discriminant of P is -23, so $P'(b)|P'(\alpha)|^2 = 23$. Also $(b+2)|\alpha+2|^2 = -P(-2) = 1$. Since $P(-2) = -1$ and $P(-7/4) = 1/64$, we have $-2 < b < -7/4$ and $P'(b) > 51/16$. The two components together add more than $\pi(51/2944)$ to the known interior area, because $2|c'(0)|^2 = P'(b)/(736(b+2)) > 51/2944$. Subtracting from (5) gives the paper bound

$$0 \le \operatorname{area}(\partial\mathcal{M}) < \pi\left(\frac{229}{800} - \frac{51}{2944}\right) < 0.85.$$

A sharper comparison using prior certified computations

Irving reports $\operatorname{area}(\mathcal{M}) \le 239397/131072$ [6]. A separate 2026 ball-arithmetic certificate gives 1.50651 as a strict lower bound on the combined area of disjoint open hyperbolic components [7]. The second number bounds the *interior*, which is essential to the subtraction. Since $\mathcal{M}$ is closed,

$$0 \le \operatorname{area}(\partial\mathcal{M}) = \operatorname{area}(\mathcal{M}) - \operatorname{area}(\operatorname{int}\mathcal{M}) < \frac{239397}{131072} - 1.50651 < 0.31995.$$

These two earlier certificates were not rerun here. The other bounds in this note are independent of them.

When a limb forces loss of boundary density

The following sufficient condition applies even without a numerical lower bound on the rotation denominators. Let $W \subset \operatorname{int} \mathcal{M}$ be a hyperbolic component and L one of its limbs. Suppose there is a univalent map $g : \mathbb{D} \to \mathbb{C}$ in the parameter plane with

$$L \subset g(B(0, 1/2)), \qquad g(\{z \in \mathbb{D} : \Re z < 0\}) \subset W. \tag{6}$$

Levin constructs maps of this form for angles he calls *deep*, by extending the inverse log-multiplier across the attachment point and rescaling its domain [8, Definition 5.1 and Theorem 5(1)]. The lemma only requires (6) at the particular component in question.

Lemma 3. A fixed fraction of an interior component

For every $x \in L$, there is a disk centered at x in which W occupies at least $1/5832$ of the area. The radius of this disk can be taken to be $4A$, where $A = |g'(0)|$.

Proof. The Koebe growth and distortion theorems on $|z| \le 1/2$ give

$$|g(z) - g(0)| \le 2A, \qquad |g'(z)| \ge \frac{4A}{27}.$$

Take $V = \{z : |z| < 1/2,\ \Re z < 0\}$. Both x and $g(V)$ are within $2A$ of $g(0)$, and $g(V) \subset W$. As g is injective,

$$\operatorname{area}\bigl(W \cap B(x, 4A)\bigr) \ge \operatorname{area}(g(V)) \ge \frac{\pi}{8}\left(\frac{4A}{27}\right)^2 = \frac{2\pi A^2}{729}.$$

Dividing by $16\pi A^2$ proves the claim. □

A second zero-area class

The set of boundary parameters belonging to limbs with (6) for infinitely many *distinct* parents has area zero. To see this, let $A_j = |g_j'(0)|$. The regions $g_j(V)$ lie in pairwise disjoint open hyperbolic components of the bounded set $\mathcal{M}$. Their areas sum to a finite number, so the lower bound above gives $A_j \to 0$. In the disks $B(x, 4A_j)$ a fixed fraction belongs to the interior. Hence x cannot be a density-one point of $\partial\mathcal{M}$. The Lebesgue density theorem proves the assertion.

This isolates a concrete next step for subexponential towers. It would suffice to obtain charts satisfying (6) at infinitely many levels for almost every remaining boundary parameter, or to prove a comparable loss of density by another method. Page 8 treats a family of small angles by controlling the whole limb with nested parameter neighborhoods.

A zero-area result at quadratic growth

Consider direct satellite towers with parent periods n_j and rotations $t_j = p_j/q_j$ as on page 1. The set of their parameters satisfying both

$$U = \sup_j n_j|t_j| < \infty, \qquad V = \limsup_{j\to\infty} q_j/n_j^2 > 0 \tag{7}$$

has area zero. For example, $|p_j| = 1$ and $n_j^2 \le q_j \le 2n_j^2$ at every sufficiently late stage satisfy (7), even though $\log q_j/n_j \to 0$.

The published neighborhoods

Use $\alpha > 0$ and Q_0 from Levin's addendum [10, Lemma 0.5, Claim 1(i)–(iv)]. Straighten a fixed prefix of period $N = n_J$ and write $\nu_m = n_{J+m}/N$. Choose J with $U/N < \alpha/(8\pi)$ and $N^2V > 8/\alpha$, and with all subsequent $q_j > Q_0$. This last choice is possible since $q_j \ge n_j/U \to \infty$. In the new coordinates the tower starts at the main cardioid, $\nu_m|t_{J+m}| < \alpha/(8\pi)$, and $q_{J+m}/\nu_m^2 > 8/\alpha$ infinitely often.

Under these hypotheses the cited claim supplies nested parameter neighborhoods R_m, $m \ge 1$, determined by the first m rotations. They satisfy

$$\operatorname{diam} R_m \le C\, 2^m/\nu_m^3 \tag{8}$$

with an absolute C in these main-cardioid coordinates. Indeed, the inverse log-multiplier is univalent on a disk of radius $\alpha/(2\nu_m)$ and its derivative there is bounded by $C_0 2^m/\nu_m^2$. Claim 1(iv) puts the entire mth limb in R_m whenever $q_{J+m}/\nu_m^2 > 8/\alpha$. Hence a parameter in the tower belongs to R_m for infinitely many m and, by nesting, to their closures at every level. This inclusion is why the second part of (7) matters.

Counting all possible continuations

Fix $s > 2/3$. Choose $Q' \ge 2$ so large that $\theta = 2^s \sum_{q\ge Q'} q^{1-3s} < 1$. Split the normalized family into countably many fixed prefixes after which every denominator is at least Q'. For one such prefix, there are at most q reduced rotations of denominator q. Each new rotation multiplies the bound in (8) by $2/q^3$ because it multiplies the parent period by q. For ℓ additional levels the sum of the sth powers of all cover diameters is therefore at most a prefix-dependent constant times θ^ℓ. The maximum diameter tends to zero as well. This proves Hausdorff dimension at most $2/3$ for the normalized class. If the numerators are bounded by P, use $2P$ choices per denominator to obtain dimension at most $1/3$.

Both bounds imply zero area in those coordinates. Pulling back over the countably many choices of normalized copy preserves nullity by page 4. In particular this controls whole families of parameters with polynomial period multipliers. The example $q_j = j + 2$ on page 4 still escapes both growth criteria.

A criterion using successive denominators

There is another family for which the cover on page 8 gives area zero. Suppose that from some stage onward

$$p_j \in \{-1, 1\}, \qquad q_{j+1} \geq q_j^2. \tag{9}$$

The second inequality compares consecutive rotation denominators. It imposes no separate quadratic threshold in the full parent period n_j.

Proof. Straighten a fixed prefix beginning sufficiently late that (9) holds and the first remaining denominator exceeds all the fixed thresholds below. Denote the tail denominators by d_m, and put $\nu_0 = 1$, $\nu_{m+1} = \nu_m d_m$. Then

$$\frac{\nu_{m+1}}{d_{m+1}} = \frac{\nu_m d_m}{d_{m+1}} \leq \frac{\nu_m}{d_m} \leq \frac{1}{d_0}.$$

Thus Levin's small-angle hypothesis on page 8 holds uniformly once $1/d_0 < \alpha/(8\pi)$. Claim 1(iii) supplies shrinking, nested neighborhoods R_m containing their respective attachment points c_m. It follows that the common limit of the attachment points belongs to every $\overline{R_m}$. This step uses the first condition of (7) alone.

To identify that limit with the parameter of an actual tower, apply Cheraghi and Shishikura's Theorem D [11]. For the signed unit numerators in (9) and d_0 large, it bounds the diameter of the parameters realizing the first m satellite renormalizations by $C_1\lambda^m$ for a fixed $0 < \lambda < 1$. A parameter of the tower and the later attachment points lie in the closure of each such finite-prefix locus. Indeed, the later components belong to that nested copy and their roots are limits of parameters in the components. As the prefix diameters tend to zero, the parameter equals the limit of the attachment points.

Now the bounded-numerator covering sum on page 8 applies with $P = 1$ to these $\overline{R_m}$. The normalized class has Hausdorff dimension at most $1/3$. Countably many inverse straightenings carry it back to a set of area zero in the original plane. □

The exact-squaring subfamily has dimension zero

If $d_{m+1} = d_m^2$ throughout the tail and d_0 is fixed, only the signs can vary. At level m there are at most 2^m possible prefixes, while $\nu_m = d_0^{2^m-1}$. By (8), for every $s > 0$ the cover cost is at most

$$2^m \left(C\, 2^m d_0^{-3(2^m-1)} \right)^s \longrightarrow 0.$$

Quasiconformal maps are locally Hölder, so they preserve Hausdorff dimension zero. Counting initial denominators and finite prefixes, and exhausting away from roots as on page 4, preserves the conclusion. Here the period multiplier is only linear in the normalized parent period, since $d_m = d_0\nu_m$. This family is outside both the exponential criterion and the second condition in (7).

SOURCES

References and proof dependencies

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[2] G. Levin, *Rigidity and non-local connectivity of Julia sets of some quadratic polynomials*, Comm. Math. Phys. 304 (2011), 295–328. Eq. (8) is the Yoccoz disk estimate. Theorems 5–6 give the two diameter bounds. Section 3.2 gives the limb correspondence and period rescaling under straightening. The linked version includes the later addendum. Source.

[3] M. Lyubich, *Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture*, Ann. of Math. 149 (1999), 319–420. Theorem 5.5 and the immediately following remark give quasiconformal equivalence for primitive copies and away from the roots of satellite copies. Source.

[4] J. Milnor, *Periodic orbits, external rays and the Mandelbrot set*, Astérisque 261 (2000), 277–333. Gives the multiplier parametrization of hyperbolic components. Source.

[5] D. Bittner, L. Cheong, D. Gates and H. D. Nguyen, *New approximations for the area of the Mandelbrot set*, Involve 10 (2017), 555–572, §1. Gives the exterior area formula and the Faber identities of Ewing and Schober used on page 6. Source.

[6] G. Irving, *Formalised Mandelbrot renders*, area estimates at depth 12. The notation 239397p−17 in the report means the exact dyadic rational $239397 \cdot 2^{-17}$. Source.

[7] *A proven lower bound on the area of the Mandelbrot set*, ScienceJournal.ai research bundle (2026), claim C2 and its verification record. C2 supplies the sum for hyperbolic components used on page 6. Source.

[8] G. Levin, *Multipliers of periodic orbits of quadratic polynomials and the parameter plane*, Israel J. Math. 170 (2009), 285–315. Definition 5.1, Proposition 3 and Theorem 5(1) give the extended inverse multiplier charts used in (6). Source.

[9] D. Giarrusso and Y. Fisher, *A parameterization of the period 3 hyperbolic components of the Mandelbrot set*, Proc. Amer. Math. Soc. 123 (1995), 3731–3737. Section 2.1 gives the cubic multiplier equation used on page 6. Source.

[10] G. Levin, *Addendum to* Rigidity and non-local connectivity of Julia sets of some quadratic polynomials, Comm. Math. Phys. 325 (2014), 1171–1178. See Lemma 0.5 and Claim 1 in its proof in the combined preprint, PDF pp. 47–49 (1-based). These give the uniform neighborhoods, their diameter bound, and the whole-limb inclusion used on page 8. Source.

[11] D. Cheraghi and M. Shishikura, *Satellite renormalisation of quadratic polynomials*, Theorem D and the signed-unit-numerator example immediately after it. The estimate concerns the diameter of a parameter locus specified by a finite satellite itinerary. Source.

Status of this note

The limb estimates, the straightening theorem, and the exterior area formula are published inputs. The period-wise cover, its application to exponential growth, the dimension estimate from Levin's nested neighborhoods, and the annulus bound are the deductions presented here for mathematical review. The global zero-area conjecture remains open.